Regional Mathematical Olympiad-2008

Time: 3 hours November 09, 2008

Instructions:

- **Calculators (in any form) and protractors are forbidden.**
- **Rulers and compasses are allowed.**
- **Answer all the questions. Maximum marks: 100.**
- **Answer to each question should start on a new page. Clearly indicate the question number.**

1. Let ABC be an acute-angled triangle; let D, F be the mid-points of BC, AB respectively. Let the perpendicular from F to AC and the perpendicular from B to BC meet in N. Prove that ND is equal to the circum-radius of ABC. [15]

2. Prove that there exist two infinite sequences $\langle a_n \rangle_{n\geq 1}$ and $\langle b_n \rangle_{n\geq 1}$ of positive integers such that the following conditions hold simultaneously:

   (i) $1 < a_1 < a_2 < a_3 < \cdots$;

   (ii) $a_n < b_n < a_n^2$, for all $n \geq 1$;

   (iii) $a_n - 1$ divides $b_n - 1$, for all $n \geq 1$;

   (iv) $a_n^2 - 1$ divides $b_n^2 - 1$, for all $n \geq 1$.

   [19]

3. Suppose a and b are real numbers such that the roots of the cubic equation $ax^3 - x^2 + bx - 1 = 0$ are all positive real numbers. Prove that:

$$\text{(i)}\ \ 0 < 3ab \leq 1 \quad \text{and} \quad \text{(ii)}\ \ b \geq \sqrt{3}.$$

   [19]

4. Find the number of all 6-digit natural numbers such that the sum of their digits is 10 and each of the digits 0,1,2,3 occurs at least once in them. [14]

5. Three nonzero real numbers a, b, c are said to be in harmonic progression if $\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$. Find all three-term harmonic progressions a, b, c of strictly increasing positive integers in which $a = 20$ and b divides c. [17]

6. Find the number of all integer-sided *isosceles obtuse-angled* triangles with perimeter 2008. [16]

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