Regional Mathematical Olympiad-2007

Time: 3 hours **October 7, 2007**

Instructions:

- **Calculators (in any form) and protractors are forbidden.**
- **Rulers and compasses are allowed.**
- **Answer all the questions. Maximum total marks is 100.**
- **Answer to each question should start on a new page. Clearly indicate the question number.**

1. Let ABC be an acute-angled triangle; AD be the bisector of $\angle BAC$ with D on BC; and BE be the altitude from B on AC. Show that $\angle CED > 45°$. [17]

2. Let a, b, c be three natural numbers such that $a < b < c$ and $\gcd(c-a, c-b) = 1$. Suppose there exists an integer d such that $a+d, b+d, c+d$ form the sides of a right-angled triangle. Prove that there exist integers l, m such that $c+d = l^2+m^2$. [17]

3. Find all pairs (a, b) of real numbers such that whenever α is a root of $x^2+ax+b=0$, α^2-2 is also a root of the equation. [17]

4. How many 6-digit numbers are there such that:

   (a) the digits of each number are all from the set $\{1, 2, 3, 4, 5\}$;

   (b) any digit that appears in the number appears at least twice?

   (Example: 225252 is an admissible number, while 222133 is not.) [16]

5. A trapezium $ABCD$, in which AB is parallel to CD, is inscribed in a circle with centre O. Suppose the diagonals AC and BD of the trapezium intersect at M, and $OM = 2$.

   (a) If $\angle AMB$ is $60°$, find, with proof, the difference between the lengths of the parallel sides.

   (b) If $\angle AMD$ is $60°$, find, with proof, the difference between the lengths of the parallel sides.

   [17]

6. Prove that:

   (a) $5 < \sqrt{5} + \sqrt[3]{5} + \sqrt[4]{5}$;

   (b) $8 > \sqrt{8} + \sqrt[3]{8} + \sqrt[4]{8}$;

   (c) $n > \sqrt{n} + \sqrt[3]{n} + \sqrt[4]{n}$ for all integers $n \geq 9$.

   [16]